EXAMPLES AND COUNTEREXAMPLES FOR PERLES' CONJECTURE

CHRISTIAN HAASE AND GÜNTER M. ZIEGLER*

Abstract. The combinatorial structure of a d-dimensional simple convex polytope – as given, for example, by the set of the $(d-1)$-regular subgraphs of the facets – can be reconstructed from its abstract graph. However, no polynomial/efficient algorithm is known for this task, although a polynomially checkable certificate for the correct reconstruction exists.

A much stronger certificate would be given by the following characterization of the facet subgraphs, conjectured by M. Perles: "*The facet subgraphs of a simple d-polytope are exactly all the $(d-1)$-regular, connected, induced, non-separating subgraphs.*"

We present non-trivial classes of examples for the validity of Perles' conjecture: In particular, it holds for the duals of cyclic polytopes, and for the duals of stacked polytopes.

On the other hand, we observe that for any 4-dimensional counterexample, the boundary of the (simplicial) dual polytope P^{Δ} contains a 2-complex without a free edge, and without 2-dimensional homology. Examples of such complexes are known; we use a modification of "Bing's house" (two walls removed) to construct explicit 4-dimensional counterexamples to Perles' conjecture.

1. Introduction

If P is a d-dimensional simple polytope, then its graph $G = G(P)$ is a d-regular, d-connected graph. If F is any facet of P, then $G_F = G(F)$ is a $(d-1)$-regular, induced, connected, non-separating subgraph of P. Whether these properties already characterize the subgraphs of facets of a simple polytope was asked by Perles [19] a long time ago:

Does every $(d-1)$-regular, induced, connected, non-separating subgraph of the graph of a simple d-polytope correspond to a facet of P?

This question is important in the context of reconstruction of polytopes. (See [15] for a general source on reconstruction of polytopes, and in particular for Perles' unpublished work in this field.) Simple polytopes can theoretically be reconstructed from their graphs – this was originally conjectured by Perles, and first proved by Blind & Mani [4], and particularly elegantly by Kalai [14]. However, although Kalai's proof can be implemented to reconstruct polytopes of reasonable size [1], the reconstruction is far from easy or efficient (in the theoretical or practical sense).

Using ideas from Kalai's paper [14], Joswig, Kaibel & Körner [13] derived a certificate for the reconstruction of a simple polytope that can indeed be checked in polynomial time. They do, however, not (yet) have a polynomial time algorithm to find/construct such a certificate. If Perles' question had a positive answer, then the reconstruction would be much easier, as was noted by Kalai [14], and by Achatz & Kleinschmidt [1]. In particular, Perles' conjecture proposes an efficient criterion for recognizing the facet subgraphs of simple polytopes, and thus an easy check whether a given list of subgraphs is indeed the complete list of facet subgraphs.

Here we first establish that Perles' conjecture is true for non-trivial classes of polytopes, including all the duals of stacked polytopes, and the duals of cyclic polytopes (Section 3).

However, our main result is that, perhaps surprisingly, Perles' conjecture is *not true* in general. To prove this, we first identify certain topological obstructions for Perles' conjecture (Section 4). We then take (known) 2-dimensional complexes that realize this obstruction – the simplest one to handle being known as "Bing's house with two rooms" (with two walls removed). We describe fairly general methods to produce counterexamples from such 2-complexes. A small, explicit counterexample derived this way will be presented as an electronic geometry model (`www.eg-models.de`).

*Supported by Deutsche Forschungsgemeinschaft (DFG).

2. Versions of Perles' Conjecture

We will distinguish two slightly different versions of Perles' conjecture, where the first and stronger version is the one stated in [14], see also [22, Problem 3.13*].

Definition 1. A simple d-polytope P *satisfies Perles' conjecture* if every induced, connected, $(d-1)$-regular, non-separating subgraph of $G(P)$ is the graph of a facet of P.
A simple d-polytope P *weakly satisfies Perles' conjecture* if every induced, $(d-1)$-connected, $(d-1)$-regular, non-separating subgraph of $G(P)$ is the graph of a facet of P.

If P is a simple polytope, then its polar dual P^{Δ} is simplicial. The vertices and edges of P correspond to the facets and ridges of P^{Δ}. The different notions in Perles' conjecture for P have immediate translations to the combinatorics of P^{Δ}. This yields the "puzzle" reformulation of [4].

- An induced subgraph H of the graph of P corresponds to a collection of facets and ridges of P^{Δ} such that, whenever two adjacent facets belong to the collection, then their common ridge also is in the collection. We will present this collection of facets and ridges in terms of the pure simplicial $(d-1)$-complex $\Gamma(H)$ that it generates. The dual graph of this "pseudomanifold" is the one that Perles' conjecture in the original form refers to.
- The induced subgraph H is $(d-1)$-regular if and only if every $(d-1)$-simplex in $\Gamma(H)$ has exactly one free $(d-2)$-simplex. (A simplex is *free* in a simplicial complex if it is contained in exactly one other/larger face of the complex.)
- The induced subgraph H is connected if and only if the corresponding subcomplex $\Gamma(H)$ is dually connected, that is, any two $(d-1)$-simplices can be joined by a chain of successively adjacent facets and ridges. (This is often called "strongly connected" for pseudomanifolds.)
- The induced subgraph does not separate its complement (graph theoretically) if and only if the corresponding subcomplex does not separate its complement (topologically): this is true since the topological complement retracts to the "dual block complex" [17] whose 1-skeleton is the graph of the complement.
- If H is the graph of a facet, then the corresponding subcomplex $\Gamma(H)$ is a vertex star: the collection of all facets that contain a given vertex.

Definition 2. A simplicial d-polytope *satisfies Perles' conjecture* if the vertex stars are the only pure $(d-1)$-dimensional, dually connected, non-separating subcomplexes of its boundary for which every maximal simplex has exactly one free face.

A simplicial d-polytope *weakly satisfies Perles' conjecture* if the vertex stars are the only pure $(d-1)$-dimensional, dually $(d-1)$-connected, non-separating subcomplexes of its boundary for which every maximal simplex has exactly one free face.

3. Positive results

Before embarking on the construction of counterexamples, and as an indication why these need to be rather complicated, we will here demonstrate that Perles' conjecture is not only *quite plausible*, but also that it is *true* on large classes of examples: it is certainly not an "unreasonable" conjecture.

We may safely assume that Perles himself verified his conjecture for non-trivial classes of examples; the Diplomarbeit of Stolletz [20] also has a number of positive results. Thus, Perles' conjecture is true, for example, for all d-dimensional polytopes with $d \leq 3$ (even without a restriction to simple/simplicial polytopes), for all 4-dimensional simple polytopes with at most 8 vertices [20, Sect. 5.2.1], for all 4-dimensional product polytopes [20, Kap. 2], for all 4-dimensional cyclic polytopes [20, Kap. 4], and for all stacked polytopes [20, Satz 3.1.3].

Indeed, the treatment of vertex truncations, and thus of stacked polytopes, presents no greater difficulties.

Proposition 1 ([20, Lemma 3.1.2])**.** *Truncation ("cutting off a simple vertex") preserves the validity of Perles' conjecture: A vertex-truncated simple d-polytope $P' := \tau_v(P)$ satisfies Perles' conjecture if and only if P satisfies it.*

Corollary 2 (Stolletz [20, Satz 3.1.3]). *All stacked polytopes (dually: the multiple vertex-truncations of simplices) satisfy Perles' conjecture.*

We now develop a rather systematic method to prove Perles' conjecture for classes of simple polytopes. We demonstrate its use for products, for wedges, and in particular for the duals of cyclic polytopes. Here is the idea: Let P be a simple d-polytope, $G = G(P)$ its graph, with vertex set V. We are interested in $(d-1)$-regular induced connected subgraphs H that don't separate; as induced subgraphs they are given by their vertex sets $V_H \subset V$. The 'Ansatz' is to concentrate on the vertex set $\overline{V} = V \setminus V_H$ of the complement of H in G. In the *first step* we (try to) classify all those vertex sets $\overline{V} \subset V$ for which

(1) the induced subgraph $G[\overline{V}]$ is connected, and
(2) $\overline{V}$ is triangle-closed and quadrilateral-closed: if $\overline{V}$ contains "all but one" vertices of a triangle or quadrilateral of G, then it must contain all its vertices.

The first condition is necessary since we want H to be non-separating. If the second condition is violated, then the "missing vertex" has two neighbors in $\overline{V}$, and thus has degree at most $d-2$ in H.

In the *second step* we then identify those sets $\overline{V}$ for which every vertex not in $\overline{V}$ has exactly one neighbor in $\overline{V}$. Our hope is then to end up with only the (complements of) facet subgraphs, plus the trivial examples given by $\overline{V} = V$.

Proposition 3. *If two simple polytopes P_1 and P_2 satisfy Perles' conjecture, then so does their product $P_1 \times P_2$.*

Proof. Let $H \subset G(P_1 \times P_2) = G_1 \times G_2$ be a $(d_1 + d_2 - 1)$-regular subgraph that is connected and does not separate ($d = d_1 + d_2$). Now (Step 1) if the complement vertex set $\overline{V}$ satisfies (1) and (2), then it is a product set, $\overline{V} = \overline{V}_1 \times \overline{V}_2$.

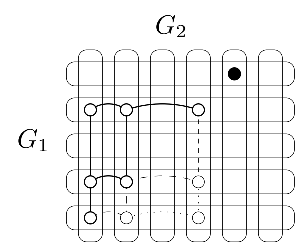

Figure 1: Property (2) at work in a product graph

Now (Step 2) if we can choose $v_i \in V_i \setminus \overline{V}_i$ for $i = 1, 2$, then this yields a vertex $(v_1, v_2) \in H$ that in $G_1 \times G_2$ has graph-theoretic distance at least 2 from the complement set $\overline{V}_1 \times \overline{V}_2$ (symbolized by the vertex $\bullet$ in our figure). Thus this vertex of H has degree $d_1 + d_2$, which is impossible.

Hence we get that the $(d_1 + d_2 - 1)$-regular, induced, connected and non-separating subgraphs all have the form $H = G_1 \times H_2$ or $H = H_1 \times G_2$, where $H_i \subset G_i$ is $(d_i - 1)$-regular, induced, connected and non-separating. By Perles' conjecture, which is valid for P_i, it follows that H_i is the graph of a facet of P_i, and hence that H is the graph of a facet of $P_1 \times P_2$. □

We refer to Klee & Walkup [16] and to Holt & Klee [11] for the construction of the wedge $\omega_F P$ of a polytope P over a facet F. (It is what you think it ought to be.)

Proposition 4. *If P is a simple d-polytope that satisfies Perles' conjecture, and F is a facet of P, then $\omega_F(P)$ satisfies Perles' conjecture (and conversely).*

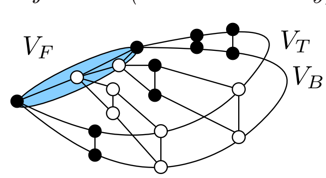

Figure 2

Sketch of proof. The vertex set of $P' := \omega_F(P)$ may be decomposed as $V_F \cup V_T \cup V_B$, into the vertices that lie in the face F resp. not in the bottom facet resp. not in the top facet of the wedge. In Step 1 one then verifies that if $\overline{V}$ meets both V_B and V_T, then it contains always "none or both" from a pair of corresponding top and bottom vertices. □

Lemma 5. *If $d > 2$ is odd, then $C_d(n)^\Delta$ is combinatorially equivalent to $\omega_F C_{d-1}(n-1)^\Delta$, for a facet $F \cong C_{d-2}(n-2)^\Delta$. If $d > 1$ is even, and $n = d+1$, then $C_d(n)^\Delta$ is a d-simplex; for $n = d+2$ it is a product of two $\frac{d}{2}$-simplices.*

Proof. To be derived from Gale's evenness criterion; see [9, p. 62] or [22, p. 14]. □

Theorem 6. *The cyclic polytopes $C_d(n)$ satisfy Perles' conjecture.*

Proof. We work with the duals, $P = C_d(n)^\Delta$. By Propositions 3 and 4 plus Lemma 5, we need only treat the case where $d = 2e$ is even, and $n > d+2$.

(i.) The graph $G = G(C_d(n)^\Delta)$ has a simple combinatorial description, via Gale's evenness criterion: Its vertices v correspond to those subsets $S \in \binom{[n]}{d}$ which split into a union of e adjacent pairs modulo n (that is, we identify the elements of $[n] = \{1, \ldots, n\}$ with $\mathbb{Z}_n$). The splitting into adjacent pairs is always unique. We shall call a "block" any non-empty union of adjacent pairs that is contiguous, that is, without a gap. Two vertices are adjacent if they differ in a single element, that is, if one arises from the other by moving one block by "one unit." This also provides a canonical orientation on each edge: we put a directed edge $v_S \twoheadrightarrow v_T$ if we get from S to T by moving a block "up" (mod n). The resulting digraph is, of course, not acyclic.

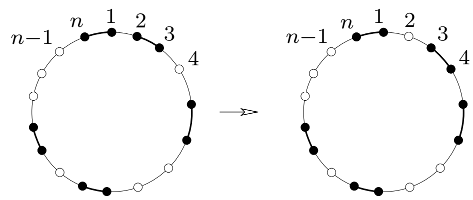

Figure 3: One single pair is moved, thus we get an edge of G, and of G', corresponding to 12356... → 13456....

We shall also consider the subgraph $G' \subseteq G$, which has the same vertex set as G, but only retains those directed edges that correspond to moves of single pairs. (In Figure 4 below, this is the graph formed by the straight edges only.) The structure of the digraph G' is closely linked to the poset

$$L \;\; := \;\; \{(j_1, j_2, \ldots, j_d) \in \mathbb{Z}^d : j_{2k} = j_{2k-1} + 1,\ j_{2k+1} \geq j_{2k} + 1,\ j_d \leq j_1 + n - 1\},$$

equipped with componentwise partial order. This poset is a distributive lattice. Moreover, reduction modulo n defines a surjective, and locally injective, digraph map $\Phi : L \longrightarrow G'$, from the Hasse diagram of $L_d(n)$ onto the digraph G'.

(ii.) Now assume (Step 1) that $\overline{V} \subset V$ satisfies (1) and (2). Every move of a block can be decomposed into a sequence of moves of pairs. Thus property (2) implies that also $G'[\overline{V}]$ is connected (as an undirected graph): every directed arc in $G[\overline{V}]$ canonically corresponds to a sequence of directed arcs in $G'[\overline{V}]$. Thus $G[\overline{V}]$ is acyclic if and only if $G'[\overline{V}]$ is acyclic.

(iii.) Next we treat the case that $G'[\overline{V}]$ contains a directed cycle. Every such cycle lifts into a two-way infinite, maximal chain $C \subseteq \Phi^{-1}(\overline{V})$ in the lattice L. *Every* element $w \in L$ is contained in a finite interval $[v', v'']$ between elements $v', v'' \in C$. Now C restricts to a maximal chain γ_0 in the interval $[v', v'']$, while w will lie on some other maximal chain γ of this interval. But in a distributive (and hence semimodular) lattice one can move from any maximal chain to any other one by one-element exchanges, see Björner [3, Sect. 6]. This implies via (2) not only the elements of γ_0, but also all elements of chains that we can move to, belong to $\Phi^{-1}(\overline{V})$. Thus we get $w \in \Phi^{-1}(\overline{V}) = L$, and hence $\overline{V} = V$.

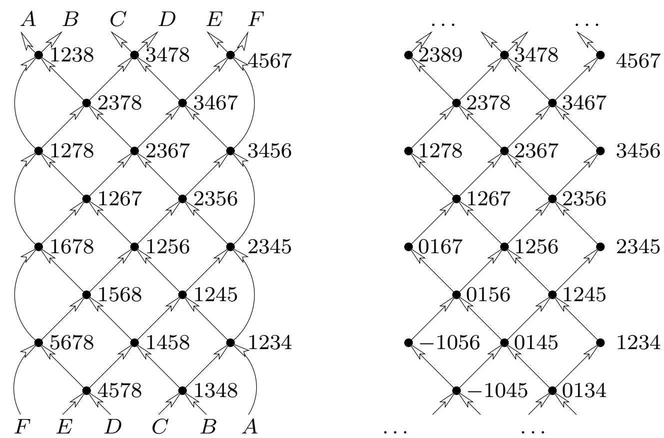

Figure 4: This depicts the digraph G, and the lattice L, for $d = 4$, $n = 8$. The digraph G is finite, the ends are to be identified according to the capital letters. It is not planar, but for $d = 4$ it embeds into a Möbius band. The lattice L is infinite.

(iv.) Thus we may assume that $G'[\overline{V}]$ is acyclic. We can then identify $G[\overline{V}]$ with an induced subgraph of the Hasse diagram of L: every connected component of $\Phi^{-1}(\overline{V})$ is isomorphic to $G'[\overline{V}]$. With this the properties (1) and (2), and lower and upper semi-modularity of the lattice L, imply that $\overline{V}$ corresponds to an interval in L (see again [3]); that is, there are elements $x_0, x_1 \in L$ such that $G'[\overline{V}]$ is isomorphic to the Hasse diagram of the interval $[x_0, x_1] \subseteq L$. This corresponds to unique vertices $v_0, v_1 \in \overline{V}$ such that $\overline{V}$ consists of all vertices of G' that lie on a directed path of minimal length from v_0 to v_1.

(v.) Assume that $\overline{V}$ contains no "no-gap vertex," whose set S consists of one single block of size d. (In the case $d = 4$, cf. Figure 4, this means that $\overline{V}$ contains no vertex on the border of the Möbius strip.) Then every no-gap vertex needs to have exactly one neighbor in $\overline{V}$, which is a one-gap vertex consisting of d elements from a block of $d+1$ adjacent vertices. Every such block corresponds to a $(\frac{d}{2}+1)$-clique; these cliques contain two no-gap vertices each, and each no-gap vertex is contained in two of these $(\frac{d}{2}+1)$-cliques. (In Figure 4, this corresponds to the chain of triangles in the boundary of the Möbius strip.)

Now we lift the situation to the lattice L, where the chain of cliques gives rise to $\frac{d}{2}$ distinct chains, which are disjoint by our assumption $n > d+2$. The interval $[x_0, x_1]$ can contain at most one element from each of the chains. Thus it can contain at most $\frac{d}{2}$ elements from the chains. Thus at most d no-gap vertices are adjacent to a vertex in $[x_0, x_1]$. Projecting this back to G, we find that at most d no-gap vertices are adjacent to a vertex in $\overline{V}$. But they all have to be, so we get $n \leq d$, a contradiction.

(vi.) If $\overline{V}$ contains a one-block vertex, then we see, using (2), that both v_0 and v_1 must be one-block vertices. By symmetry, we may then assume that $S_0 = \{1, 2, \ldots, d\}$ and $S_1 = \{k+1, \ldots, d+k-1\}$ for some k. This ends Step 1: and Step 2 – the identification of those parameters k for which each vertex in $V \setminus \overline{V}$ has exactly one neighbor in $\overline{V}$ – is now easy. □

4. The obstruction

We now restrict our discussion to the case $d = 4$, where we will construct a counterexample to the weak Perles conjecture.* From it, one gets counterexamples to the weak Perles conjecture in all dimensions $d > 4$, from the constructions of Section 3, such as wedges and products.

The discussion in this section motivates our construction; it leads us to well-characterized obstructions to the validity of Perles' conjecture (for any specific polytope). For simplicity, we formulate this for $d = 4$; the generalization to $d \geq 4$ is immediate.

*The pictures try to illustrate the situation in $d = 3$, though.

Proposition 7. *Let P^Δ be a simplicial 4-polytope whose boundary complex $\Delta := \Delta(\partial P^\Delta)$ contains a dually connected pure 3-dimensional subcomplex Γ, all whose tetrahedra have exactly one free triangle in Γ.*

Then Γ (and in particular Δ) contains a pure 2-dimensional, dually connected subcomplex $\mathrm{core}(\Gamma)$ *without free edges. Moreover, Γ separates the boundary of P^Δ if and only if* $\mathrm{core}(\Gamma)$ *does, and* $\mathrm{core}(\Gamma)$ *is empty if and only if Γ is a vertex star.*

In the case $d = 3$, $\mathrm{core}(\Gamma)$ would be 1-dimensional, with neither a cycle (non-separating) nor a free face: a leafless tree. This (re)proves Perles' conjecture for 3-polytopes.

Proof. We specify a pure 2-dimensional subcomplex $\mathrm{core}(\Gamma)$ of Γ which will comply with our conditions (the definition is due to Carsten Schultz): *Every tetrahedron $\sigma \in \Gamma$ has a unique vertex $v(\sigma)$ opposite to its free face. A triangle τ belongs to* $\mathrm{core}(\Gamma)$ *if both its neighboring tetrahedra σ_1, σ_2 belong to Γ, and if $v(\sigma_1) \neq v(\sigma_2)$.*

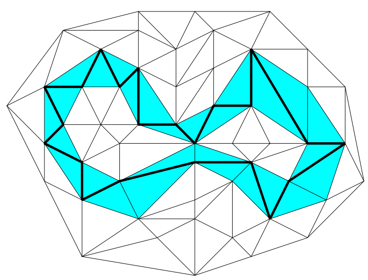

Figure 5: If the grey 2-complex is Γ, then the black 1-dimensional subcomplex is its core.

Claim 1. The complex $\mathrm{core}(\Gamma)$ has no free edge.

Suppose that ϱ is a free edge of the triangle $\tau \in \mathrm{core}(\Gamma)$. The tetrahedra of Δ which contain ϱ are cyclically ordered: $\sigma_1, \dots, \sigma_n$, as in Figure 6.

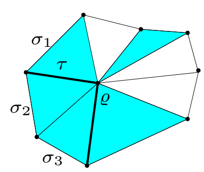

Figure 6

We can assume that $\tau = \sigma_1 \cap \sigma_2$ and $\sigma_3 \in \Gamma$. For, if σ_3 and σ_n were both not in Γ, then this would imply $v(\sigma_1) = v(\sigma_2)$, and hence $\tau \notin \mathrm{core}(\Gamma)$.

Because $\sigma_2 \cap \sigma_3 \notin \mathrm{core}(\Gamma)$, we have $v(\sigma_3) = v(\sigma_2)$. This vertex lies in $\sigma_2 \cap \sigma_3$, and it cannot be the vertex of σ_2 that is opposite to τ; thus we get that $v(\sigma_3) = v(\sigma_2) \in \varrho$. In particular, this means that $\sigma_3 \cap \sigma_4$ is not the free face of σ_3, and thus $\sigma_4 \in \Gamma$. Iterating these arguments, we see that $\sigma_1, \dots, \sigma_n \in \Gamma$ and $v(\sigma_n) = \ldots = v(\sigma_2) \neq v(\sigma_1)$. So, $\sigma_n \cap \sigma_1$ is another triangle in $\mathrm{core}(\Gamma)$.

Claim 2. The complex $\mathrm{core}(\Gamma)$ is empty if and only if Γ is a vertex star.

If $\mathrm{core}(\Gamma)$ is empty, as Γ is dually connected, all the vertices $v(\sigma)$ are equal, say, v_0. Hence Γ is part of $\mathrm{star}_\Delta(v_0)$. But then Γ must be the whole star, because there is only one free face per tetrahedron.

Claim 3. If $\mathrm{core}(\Gamma)$ separates then so does Γ.

Alexander duality implies that a subcomplex of a 3-sphere separates if and only if it has non-trivial 2-dimensional homology. The 3-dimensional complex Γ collapses down to a 2-dimensional subcomplex Γ', which contains $\mathrm{core}(\Gamma)$. Since Γ' has dimension 2, it is clear that the (relative)

homology group $H_3(\Gamma', \mathrm{core}(\Gamma))$ vanishes. Hence, the map ι in the exact sequence

$$\begin{array}{ccccccc} H_3(\Gamma, \mathrm{core}(\Gamma)) & \longrightarrow & H_2(\mathrm{core}(\Gamma)) & \xrightarrow{\iota} & H_2(\Gamma) & \longrightarrow & \cdots \\ \| & & & & & & \\ H_3(\Gamma', \mathrm{core}(\Gamma)) = 0 & & & & & & \end{array}$$

is injective, and the claim is proved. □

Let us further analyze the above situation. The aim is to reveal the combinatorial properties of complexes that appear as a core (in addition to the topological ones that we have already seen). In Section 5 we will show how to guarantee these properties using stellar subdivisions.

Consider now the structure of Γ locally, restricted to the (open) star of a vertex v_0. This vertex star is divided by $\mathrm{core}(\Gamma)$ into several pieces and Γ is a union of some of these pieces:

Lemma 8. *Let $\sigma, \sigma' \in \mathrm{star}_\Delta(v_0)$ be tetrahedra in the same piece, that is, such that there is a dual path in $\mathrm{star}_\Delta(v_0)$ between σ and σ' that does not meet $\mathrm{core}(\Gamma)$. If $\sigma \in \Gamma$ and $v(\sigma) = v_0$, then $\sigma' \in \Gamma$ and $v(\sigma') = v_0$.*

Corollary 9. *For every vertex v_0 of $\mathrm{core}(\Gamma)$ the complex which is generated by $\{\sigma \in \Gamma : v(\sigma) = v_0\}$ is the closure of a union of components of $\mathrm{star}_\Delta(v_0) \setminus \mathrm{core}(\Gamma)$.*

Furthermore, two such components that correspond to different v_0's intersect at most in codimension one, and if so, then this intersection is included in $\mathrm{core}(\Gamma)$.

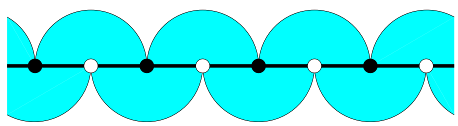

Figure 7: illustrates Corollary 9

5. A counterexample

Proposition 7 makes one ask for a 2-dimensional simplicial complex Γ, without a free edge (every edge is contained in at least two triangles), which does not separate (that is, $H_2(\Gamma, \mathbb{Z}) = \{0\}$), and which is embeddable in $\mathbb{R}^3$ (in particular, $H_1(\Gamma, \mathbb{Z})$ has no torsion). Such 2-dimensional complexes do exist. The most prominent examples are probably Borsuk's "dunce hat" [6, 21], and Bing's "house with two rooms" [2] [10, p. 4], which are even contractible.

To keep the constructions for the following simpler, we shall put an extra condition on the complexes we look at: we want them to be "essentially manifolds," that is, we want every point to have a closed neighborhood that is homeomorphic either to two or to three triangles that are joined together at a common edge. (Thus, we admit singular curves, along which a manifold branches into three parts, but we do not admit singular points that may be more complicated than the points on a singular curve.)

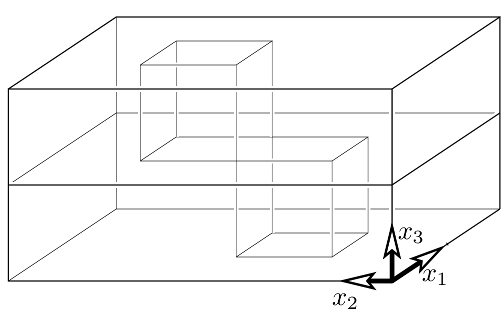

Figure 8: The complex $\mathcal{B}$ has two rooms: The downstairs room is connected to the outside via the chimney through the upstairs room, the upstairs room is connected to the outside via the chimney through the downstairs room.

In the following, we provide an as-concrete-as-possible description of a specific counterexample of this type. Further counterexamples of the same type may be obtained along the same lines.

Our point of departure is a modification $\mathcal{B}$ of Bing's house. Two extra walls are missing that would usually be added to make the interior of each room simply connected (cf. Figure 8).

This $\mathcal{B}$ can be embedded into a pile of $2 \times 3 \times 4$ cubes – a cubical complex. Triangulate the pile by the arrangement of hyperplanes $x_i - x_j = k$ to get a simplicial complex (cf. Figure 9). Add a cone over the boundary to get the boundary of a simplicial 4-polytope Q^{Δ} (any subdivision by a hyperplane arrangement is "coherent"/"regular"; cf. [22, Lecture 5]).

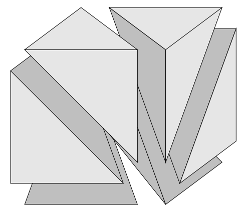

Figure 9

This polytope contains a triangulation of $\mathcal{B}$ as a subcomplex, which we also denote by $\mathcal{B}$. The vertices of this subcomplex now get "colors" 0, 1, or 2, by assigning to each vertex the sum of its coordinates modulo 3. The only edges that join vertices of the same color are the diagonals of the cubes, which are not in $\mathcal{B}$. The two chimneys look as in Figure 10.

The upper chimney:

The lower chimney:

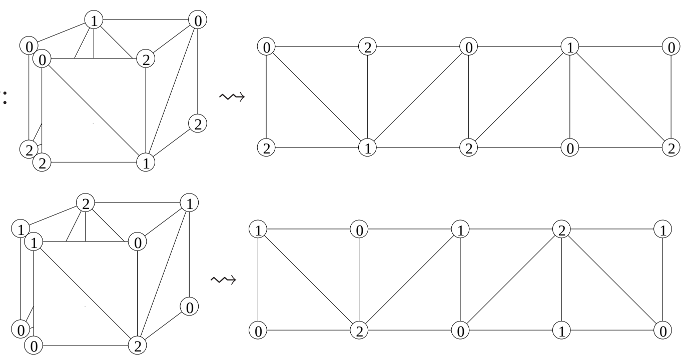

Figure 10

We now perform a sequence of stellar subdivisions on faces of Q^{Δ}, as follows:

• Perform stellar subdivisions on vertical edges in the chimneys, namely in the upper chimney on the two edges with labels $\genfrac{}{}{0pt}{}{0}{2}$, and in the lower chimney on the two edges with labels $\genfrac{}{}{0pt}{}{1}{0}$ (cf. Figure 11).

The upper chimney The lower chimney

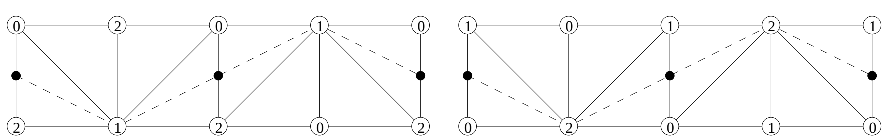

Figure 11

The resulting new edges in the chimneys (four each; dashed edges in Figure 11) are interpreted as marking the boundary between 'inside' and 'outside' in the chimneys. No color is given to these four new vertices.

With the dashed separation edges in place, we now have achieved the situation that every triangle in the core is adjacent to exactly two of "outside", "upstairs" and "downstairs". Furthermore, if a triangle is adjacent to "outside" then it has a 0-colored vertex, for "upstairs" it has a 1-colored vertex, and for "downstairs" a 2-colored vertex.

• Now perform stellar subdivisions on simplices outside $\mathcal{B}$ all whose faces belong to $\mathcal{B}$ (e. g., the diagonals of the cubes). Thus $\mathcal{B}$ becomes an induced subcomplex.

• Whenever there are vertices $v_1, v_2 \in \mathcal{B}$ of the same color, and a vertex $w \notin \mathcal{B}$ such that both

$\{w, v_1\}$ and $\{w, v_2\}$ are edges (e. g. v_1, v_2 on the boundary of the pile and w the cone vertex), perform a stellar subdivision on one of these edges.

The result of all these operations is now called P^Δ. Thus the polytope P, a counterexample to Perles' original conjecture, is obtained as its polar dual. It remains to name the vertex star pieces that we want to attach to $\mathcal{B}$ in order to get a Perles-contradicting subcomplex. For that purpose we use the vertex labels, and proceed according to the scheme

label		partial vertex star
0	$\longleftrightarrow$	outside
1	$\longleftrightarrow$	upstairs room
2	$\longleftrightarrow$	downstairs room

If a vertex with label 2 does not touch the downstairs room, then no partial vertex star is assigned, etc. The union Γ of these partial vertex stars is a non-separating subcomplex, all whose tetrahedra have exactly one free triangle opposite to 'their' vertex of $\mathcal{B}$. $\square$

6. Connectivity

Finally, we verify that our counterexample does not even weakly satisfy Perles' conjecture, that is, Γ is dually 3-connected. Our arguments will be general enough to work as well for other counterexamples of the same type (where the core is essentially a manifold). Of course, for specific counterexamples, such as the one to be presented in `www.eg-models.de`, 3-connectedness could as well be verified by explicit computer calculation.

Remove two tetrahedra $\bar{\sigma}_1, \bar{\sigma}_2$ from Γ. We want to join any two remaining tetrahedra σ_1, σ_2 by a dual path, avoiding $\bar{\sigma}_1, \bar{\sigma}_2$. This path is constructed in three steps:

1. Join σ_1, σ_2 to tetrahedra which have a full-dimensional intersection with $\operatorname{core}(\Gamma)$.
2. Join these full-dimensional intersections by a dual path in $\operatorname{core}(\Gamma)$.
3. Lift this path to a path in Γ.

Step 1 follows from a simple fact about graphs of 3-polytopes:

Lemma 10. *Every tetrahedron $\sigma \in \Gamma$ can be joined to some tetrahedron with full-dimensional intersection with* $\operatorname{core}(\Gamma)$*, avoiding $\bar{\sigma}_1, \bar{\sigma}_2$.*

Proof. The dual graph of the star of $v(\sigma)$ in Δ corresponds to a (simple) 3-dimensional polytope, and is therefore 3-connected. Any dual path form σ to a tetrahedron in a different component has to pass through $\operatorname{core}(\Gamma)$. $\square$

For Step 2, we need some preparation. In view of Step 3, we consider a graph $G^*_{\mathcal{B}}$ defined as follows: its nodes are the two-faces of $\operatorname{core}(\Gamma)$, and its edges correspond to pairs (τ_1, τ_2) of two-faces which share a one-face ϱ, and which can be joined by a dual path in $\operatorname{star}(\varrho; \Gamma)$.

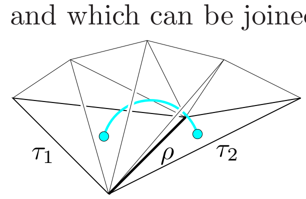

Figure 12

We are interested in the subgraphs of $G^*_{\mathcal{B}}$ that are induced by those nodes (triangles) that lie in the star of a vertex of $\operatorname{core}(\Gamma)$.

Lemma 11. *Let ϱ be a one-face of the triangle $\tau \in$* $\operatorname{core}(\Gamma)$*. Then there is at least one more triangle $\tau' \in$* $\operatorname{core}(\Gamma)$ *which contains ϱ, and is joined to τ by an edge in $G^*_{\mathcal{B}}$.*

Proof. τ is covered from both sides with tetrahedra $\sigma_\uparrow$ and $\sigma_\downarrow \in \operatorname{core}(\Gamma)$. The vertices $v(\sigma_\uparrow) \neq v(\sigma_\downarrow)$ are both vertices of τ, so that every edge of τ contains at least one of them. $\square$

Corollary 12. *For every vertex $v \in$* $\operatorname{core}(\Gamma)$*, the subgraph of $G^*_{\mathcal{B}}$ induced by the triangles incident to v is 2-connected.*

Proof. If v is a regular vertex (all incident edges have degree 2 in core(Γ)), then the corresponding local subgraph of $G^*_{\mathcal{B}}$ is a cycle: It is the dual graph of the star of v in core(Γ).

If v is a singular vertex (two incident edges have degree 3 in core(Γ)), then the subgraph of $G^*_{\mathcal{B}}$ is a union of three paths between two nodes. $\square$

Now, to verify 3-connectivity of $G^*_{\mathcal{B}}$, we use the following simple but very useful observation of Naatz:

Theorem 13 ([18, Theorem 3.3]). *A graph G on at least $k+1$ vertices is k-connected if and only if for every two vertices v and w at distance 2 there are at least k independent v-w-paths in G.*

Proposition 14. *$G^*_{\mathcal{B}}$ is 3-connected.*

Proof. Remove two nodes/triangles $\bar{\tau}_1, \bar{\tau}_2$ from $G^*_{\mathcal{B}}$, and denote τ_0, τ_∞ the two nodes which we want to join by a $\bar{\tau}_1, \bar{\tau}_2$-avoiding path. In core($\Gamma$), the intersection $\bar{\tau}_1 \cap \bar{\tau}_2$ contains at most two vertices. Now the 1-skeleton of core(Γ) is 3-connected (e. g. by Naatz' lemma). Thus there is a $\bar{\tau}_1 \cap \bar{\tau}_2$-avoiding vertex-edge-path in core(Γ) from a vertex of τ_0 to some vertex of τ_∞. That is, there are vertices $v_0, \dots, v_k$ of core(Γ) such that $v_0 \in \tau_0$, $v_k \in \tau_\infty$, and the $[v_i, v_{i-1}]$'s are edges of core(Γ).

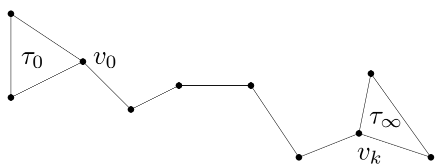

Figure 13

Choose triangles $\tau_i \in$ core(Γ) adjacent to the edges $[v_{i-1}, v_i]$.

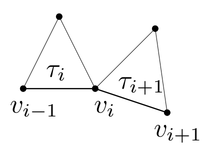

Figure 14

By Corollary 12, consecutive triangles τ_i and τ_{i+1} can be joined within the star of v_i, avoiding $\bar{\tau}_1, \bar{\tau}_2$. $\square$

Finally, we get Step 3 for free:

Corollary 15. *Γ is dually 3-connected.*

Proof. Because core(Γ) is induced, $\bar{\sigma}_i \cap$core(Γ) is a face of $\bar{\sigma}_i$. If $\bar{\sigma}_i \cap$core(Γ) is a triangle, remove the corresponding node from $G^*_{\mathcal{B}}$. If $\bar{\sigma}_i \cap$ core(Γ) is an edge, remove the edge from $G^*_{\mathcal{B}}$, which is cut by $\bar{\sigma}_i$. By Lemma 10, we can assume that we want to join two tetrahedra $\sigma_1, \sigma_2 \in \Gamma$ which intersect core(Γ) in triangles τ_1 and τ_2 respectively. By Lemma 14 we can find a path from τ_1 to τ_2 in the remaining graph. This path can be lifted to a dual path which joins σ_1 and σ_2, and which avoids $\bar{\sigma}_1, \bar{\sigma}_2$. $\square$

7. Remarks

It seems reasonable to ask whether every induced, 3-connected, 3-regular, non-separating and planar (!) subgraph of the graph of a simple 4-polytope is the graph of a facet.

Does the 120-cell satisfy Perles' conjecture?

Acknowledgements. Thanks, in particular, to Tom Braden, Włodek Kuperberg, Carsten Lange, Mark de Longueville, Carsten Schultz, Raik Stolletz and Elmar Vogt for crucial discussions.

References

[1] H. Achatz & P. Kleinschmidt: *Reconstructing a simple polytope from its graph,* in: "Polytopes – Combinatorics and Computation" (G. Kalai, G.M. Ziegler, eds.), *DMV Seminar* **29**, Birkhäuser, Basel 2000, pp. 155-165.

[2] R H Bing: *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture,* in: *Lectures on Modern Mathematics, Vol. II* (T.L. Saaty, ed.), Wiley 1964, 93-128.

[3] A. Björner: *Shellable and Cohen-Macaulay partially ordered sets,* Transactions Amer. Math. Soc. **260** (1980), 159-183.

[4] R. Blind & P. Mani-Levitska: *On puzzles and polytope isomorphisms,* Aequationes Math. **34** (1987), 287-297.

[5] G. Bohus, W. Jockusch, C. W. Lee & N. Prabhu: *On a triangulation of the 3-ball and the solid torus,* Discrete Math. **187** (1998), 259-264.

[6] K. Borsuk: *Über das Phänomen der Unzerlegbarkeit in der Polyedertopologie,* Commentarii Math. Helv. **8** (1935), 142-148.

[7] E. Gawrilow & M. Joswig: `polymake`: *A framework for analyzing convex polytopes,* in: "Polytopes – Combinatorics and Computation" (G. Kalai, G.M. Ziegler, eds.), *DMV Seminar* **29**, Birkhäuser, Basel 2000, pp. 43-73.

[8] E. Gawrilow & M. Joswig: *Polymake: A framework for analyzing convex polytopes,* 1997-2000, `http://www.math.tu-berlin.de/diskregeom/polymake/doc/`.

[9] B. Grünbaum: *Convex Polytopes,* Interscience, London 1967.

[10] A. Hatcher: *Algebraic Topology,* `http://www.math.cornell.edu/~hatcher/`, Cornell University, 2000; Cambridge University Press 2001, to appear.

[11] F. Holt & V. Klee: *Counterexamples to the strong d-step conjecture for $d \geq 5$,* Discrete Comput. Geometry **19** (1998), 33-46.

[12] M. Joswig: *Reconstructing a non-simple polytope from its graph,* in: "Polytopes – Combinatorics and Computation" (G. Kalai, G.M. Ziegler, eds.), *DMV Seminar* **29**, Birkhäuser, Basel 2000, pp. 167-176.

[13] M. Joswig, V. Kaibel & F. Körner: *On the k-systems of a simple polytope,* Preprint `math.CO:0012204`, TU Berlin, December 2000, 8 pages; *Israel J. Math.*, to appear.

[14] G. Kalai: *A simple way to tell a simple polytope from its graph,* J. Combinatorial Theory, Ser. A **49** (1988), 381-383.

[15] G. Kalai: *Some aspects of the combinatorial theory of convex polytopes,* in: "Polytopes: Abstract, Convex and Computational" (T. Bisztriczky, P. McMullen, and A. Weiss, eds.), Proc. NATO Advanced Study Institute, Toronto 1993, Kluwer Academic Publishers 1994, pp. 205-230.

[16] V. Klee & D. W. Walkup: *The d-step conjecture for polyhedra of dimension $d < 6$,* Acta Math. **117** (1967), 53-78.

[17] J. R. Munkres: *Elements of Algebraic Topology,* Addison Wesley 1984.

[18] Michael Naatz: *The graph of linear extensions revisited,* SIAM J. Disc. Math. **13** (2000), 354-369.

[19] M. Perles: *Results and problems on reconstruction of polytopes,* Jerusalem 1970, unpublished [14]; problems posed in Oberwolfach 1984 and 1986 [4].

[20] R. Stolletz: *Facetten-Graphen einfacher Polytope,* Diplomarbeit, TU Berlin 1998, 77 pages.

[21] E. C. Zeeman: *On the dunce hat,* Topology **2** (1963), 341-358.

[22] G. M. Ziegler: *Lectures on Polytopes,* Graduate Texts in Mathematics **152**, Springer-Verlag New York Berlin Heidelberg 1995; Revised edition, 1998.

[23] G. M. Ziegler: *Cyclic polytopes and their duals,* in preparation.

E-mail address: `haase@math.duke.edu`

Department of Mathematics, Duke University, Durham, NC 27708-0320, USA

E-mail address: `ziegler@math.tu-berlin.de`

MA 6-2, Institute of Mathematics, TU Berlin, D-10623 Berlin, Germany